Exercise 3

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$1 + \frac{1}{3!}x^3 - \cos x = \int_0^x (x-t)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{1 + \frac{1}{6}x^3 - \cos x\right\} = \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{gathered}\mathcal{L}\{1\} + \frac{1}{6}\mathcal{L}\{x^3\} - \mathcal{L}\{\cos x\} = \mathcal{L}\{x\}U(s)\\ \frac{1}{s} + \frac{1}{6}\left(\frac{6}{s^4}\right) - \frac{s}{s^2+1} = \frac{1}{s^2}U(s)\end{gathered}$$

Solve for $U(s)$.

$$\begin{aligned}\frac{U(s)}{s^2} &= \frac{1}{s} + \frac{1}{s^4} - \frac{s}{s^2+1}\\ U(s) &= s + \frac{1}{s^2} - \frac{s^3}{s^2+1}\\ &= \frac{1}{s^2} + \frac{s(s^2+1) - s^3}{s^2+1}\\ &= \frac{1}{s^2} + \frac{s}{s^2+1}\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}u(x) &= \mathcal{L}^{-1}\{U(s)\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\\ &= x + \cos x\end{aligned}$$